Attractors for Differential Equations with Variable Delays

Tomás Caraballo, José A. Langa

Dpto. Ecuaciones Diferenciales y Análisis Numérico. Universidad de Sevilla. Apdo. de Correos 1160. 41080-Sevilla (Spain)
E-mail: caraball@numer.us.es ; langa@numer.us.es

and

James C. Robinson

Mathematics Institute. University of Warwick. Coventry, CV4 7AL. UK
E-mail: jcr@maths.warwick.ac.uk

Using the relatively new concept of a pullback attractor, we present some results on the existence of attractors for differential equations with variable delay. We give a variety of examples to which our result applies.

1. INTRODUCTION

The theory of global attractors for autonomous systems as developed by Hale in [7] owes much to examples arising in the study of retarded functional differential equations [8] (for slightly different approaches see Babin and Vishik [1], Ladyzhenskaya [12], or Temam [15]). Although the classical theory can be extended in a relatively straightforward manner to deal with time-periodic equations, general non-autonomous equations such as

$$\dot{x}(t) = F(t, x(t), x(t - \rho(t))) \tag{1}$$

fall outside its scope.

Recently, a theory of 'pullback attractors' has been developed (see section 2) which allows many of the ideas for the autonomous theory to be extended to deal with such examples. However, until now this has only been applied to ordinary and partial differential equations.

It is our intention here to show how pullback attractors can be used to investigate the behaviour of non-autonomous delay equations. In particular, we are able to compare the dynamics of systems of ordinary differential equations with that of the same system with a small delay, and show that these are 'close' in some global sense.

2. DELAY DIFFERENTIAL EQUATIONS AS DYNAMICAL SYSTEMS

We take as our canonical example of a non-autonomous delay equation a system with one, time-varying delay, $\rho(t)$ where $\rho : \mathbb{R} \to [0,h]$ is a continuous function and $h > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = F(t, x(t), x(t-\rho(t))) \qquad x_s = \psi, \quad \psi \in \mathcal{C}. \tag{2}$$

The initial condition x_s is specified in $\mathcal{C}$, the space $C^0([-h,0];\mathbb{R}^n)$ of continuous functions from $[-h,0]$ into $\mathbb{R}^n$, and, for a function $x \in C^0([-h,T];\mathbb{R}^n)$, the notation x_s denotes the function in $\mathcal{C}$ given by

$$x_s(\theta) = x(s+\theta) \qquad \text{for all} \qquad \theta \in [-h,0]$$

(and so makes sense for any $0 \leq s \leq T$).

This equation can be written in a more general framework, which allows one to consider a larger set of problems in a unified way. Rather than make the delay explicit, we write

$$f(t, x_t) = F(t, x(t), x(t-\rho(t))),$$

and so can rewrite (2) as

$$\dot{x}(t) = f(t, x_t) \qquad x_s = \psi, \ \psi \in \mathcal{C}. \tag{3}$$

In what follows we concentrate on this form of the equation, assuming that $f : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^n$ is continuous and 'a bounded map' (i.e. maps bounded sets into bounded sets).

We note here that this formulation immediately includes examples other than the single, time-varying delay of (2). For example, the integro-differential equation (see Kuang [11] for more details)

$$\dot{x}(t) = \int_{-h}^{0} g(t, s, x(t+s))\, ds$$

also fits into this framework, although we do not develop this theory here.

It is known (Hale [7]) that for any $(s,\psi) \in \mathbb{R}\times\mathcal{C}$ there exists a unique solution $x(t; s, \psi)$ for (3) defined on $[s-h, \alpha_{s,\psi})$. We assume that $\alpha_{s,\psi} = +\infty$, for all $s \in \mathbb{R}$, since we are interested in long-time behaviour of solutions. We define a solution operator $\phi(t,s)$ which gives the solution (in $\mathcal{C}$) at time t when $x_s = \psi$, via

$$\phi(t,s)\psi = x_t(\cdot; s, \psi). \tag{4}$$

3. PULLBACK ATTRACTORS

We now discuss the theory of pullback attractors, as developed in Kloeden and Stonier [9], Kloeden and Schmalfuss [10], and Crauel *et al.* [5]). As is clear above, in the case of non-autonomous differential equations the initial time is just as important as the final time, and the classical semigroup property of autonomous dynamical systems is no longer available.

Instead of a family of one time-dependent maps $\mathcal{S}(t)$ we need to use a two-parameter process $\phi(t,s)$, as introduced above in (4) (cf. Sell [14]); $\phi(t,s)\psi$ denotes the solution at time t which was equal to ψ at time s.

The semigroup property is replaced by the process composition property

$$\phi(t,s)\phi(s,r) = \phi(t,r) \qquad \text{for all} \qquad t \geq s \geq r,$$

and, obviously, the initial condition implies $\phi(s,s)=$Id. As with the semigroup composition $\mathcal{S}(t)\mathcal{S}(s) = \mathcal{S}(t+s)$, this just expresses the uniqueness of solutions.

[It is possible to present the theory within the more general framework of cocycle dynamical systems. In this case the second component of ϕ is viewed as an element of some parameter space J, so that the solution can be written as $\phi(t,p)\psi$, and a shift map $\theta_t : J \to J$ is defined so that the process composition becomes the cocycle property,

$$\phi(t+\tau,p) = \phi(t,\theta_\tau p)\phi(\tau,p).$$

We do not pursue this approach here, but note that it has proved extremely fruitful, particularly in the case of random dynamical systems. For various examples using this general setting, see Kloeden and Schmalfuss [10], or Sell [14]. For this reason, pullback attractors are often referred to as 'cocycle attractors'].

As in the standard theory of attractors, we seek an invariant attracting set. However, since the equation is non-autonomous this set also depends on time.

Definition 3.1. Let ϕ be a process on a complete metric space X. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a (global) pullback attractor for ϕ if, for all $s \in \mathbb{R}$, it satisfies

i) $\phi(t,s)\mathcal{A}(s) = \mathcal{A}(t)$ for all $t \geq s$, and
ii) $\lim_{s\to\infty} \text{dist}(\phi(t,t-s)D,$

A(t))=0$, for all bounded subsets $D$of$X.

In the definition, $\text{dist}(A,B)$ is the Hausdorff semidistance between A and B, defined as

$$\text{dist}(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b), \text{ for } A,B \subseteq X.$$

Property i) is a generalization of the invariance property for autonomous dynamical systems. The pullback attracting property ii) considers the state of the system at time t when the initial time $t-s$ goes to $-\infty$ (cf. Chepyzhov and Vishik [3])

The notion of an attractor is closely related to that of an absorbing set.

Definition 3.2. $\{B(t)\}_{t\in\mathbb{R}}$ is said to be absorbing with respect to the process ϕ if, for all $t\in\mathbb{R}$ and all $D\subset X$ bounded, there exists $T_D(t)>0$ such that for all $\tau\geq T_D(t)$

$$\phi(t,t-\tau)D\subset B(t).$$

Indeed, just as in the autonomous case, the existence of compact absorbing sets is the crucial property in order to obtain pullback attractors. For the following result see Crauel and Flandoli [4] or Schmalfuss [13].

Theorem 3.1. *Let $\phi(t,s)$ be a two-parameter process, and suppose $\phi(t,s):X\to X$ is continuous for all $t\geq s$. If there exists a family of compact absorbing sets $\{B(t)\}_{t\in\mathbb{R}}$, then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, and*

$A(t)\subset B(t)$ for all $t\in\mathbb{R}$. Furthermore,

$$\mathcal{A}(t)=$$

$$\bigcup \bigcup \bigcup_{D\subset X} {}_{\text{bounded}}\Lambda_D(t),$$ *where*

$$\Lambda_D(t)=\bigcap_{n\in}\overline{\bigcup_{s\geq n}\phi(s,t-s)D}.$$

4. ATTRACTORS FOR NON-AUTONOMOUS DELAY DIFFERENTIAL EQUATIONS

We showed above how to define the process associated with the solution of the delay differential equation

$$\dot{x}(t)=f(t,x_t)\qquad x_s=\psi \tag{5}$$

via

$$\phi(t,s)\psi=x_t(\cdot;s,\psi).$$

We now prove a simple general result on the existence of pullback attractors for this problem. Proof that the condition of the theorem holds is significantly more onerous than the proof of the theorem itself.

THEOREM 4.1. *Suppose that $\phi(t,s)$ maps bounded sets into bounded sets, and that there exists a family $\{B(t_0)\}_{t\in\mathbb{R}}$ of bounded absorbing sets for ϕ. Then there exists a pullback attractor for problem (5).*

Proof. Using theorem 3.1 it suffices to prove that there exists a family of compact absorbing sets for ϕ. For each $t_0 \in \mathbb{R}$, define

$$K(t_0) = \phi(t_0, t_0 - h)B(t_0 - h).$$

$K(t_0)$ is clearly absorbing, since for any bounded $D \subset \mathcal{C}$ we have, for $t \geq T_D(t_0) + h$, (here $T_D(t_0)$ denotes the absorption time corresponding to the family $\{B(t_0)\}_{t\in\mathbb{R}}$)

$$\begin{aligned}\phi(t_0, t_0 - t)D &= \phi(t_0, t_0 - h)\phi(t_0 - h, (t_0 - h) - (t - h))D \\ &\subset \phi(t_0, t_0 - h)B(t_0 - h) = K(t_0).\end{aligned}$$

Also, $K(t_0)$ is bounded, since ϕ maps bounded sets into bounded sets. Finally, $K(t_0)$ is a compact subset of $\mathcal{C}$. This follows using the Arzelà-Ascoli theorem, since we have just shown that $K(t_0)$ is bounded, and the equicontinuity follows since, for $\psi \in B(t_0 - h)$ and $\theta \in [-h, 0]$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\phi(t_0, t_0 - h)\psi(\theta)\right| = \left|\frac{\mathrm{d}}{\mathrm{d}\theta}x(t_0 + \theta; t_0 - h, \psi)\right| = |f(t_0 + \theta, x_{t_0+\theta}(\cdot; t_0 - h, \psi)|$$

which is bounded, using the assumption on f. ∎

4.1. The case of strong dissipativity

In this section we suppose a dissipative property for the nonlinear term of the differential equation which will lead us to the existence of a uniform (over $t \in \mathbb{R}$) bounded absorbing set for the process ϕ and hence a pullback attractor.

We will suppose in this section that for some $\alpha > 0$, $\beta \geq 0$

$$\langle f(t, \psi), \psi(0)\rangle \leq -\alpha\,|\psi(0)|^2 + \beta \quad \text{for all} \quad \psi \in \Phi(h)\mathcal{C} \tag{6}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$ and

$$\Phi(h)\mathcal{C} = \{\chi \in \mathcal{C} : \chi = \phi(s + h, s)\psi, \quad \text{some} \quad s \in \mathbb{R}, \psi \in \mathcal{C}\}.$$

(Note that $\Phi(h)\mathcal{C}$ is essentially the set of all those functions in $\mathcal{C}$ which are realisable as solutions of the equation after a time h).

Although this seems strange at a first view, note that (6) is a consequence of a more natural set of assumptions in various particular examples. Indeed, if we consider (2) with $F : \mathbb{R}^n \to \mathbb{R}^n$ uniformly bounded and uniformly continuous, i.e., for some $k \geq 0$ and some function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$,

$$|F(x)| \leq k \qquad \text{and} \qquad |F(x) - F(y)| \leq \omega(|x-y|),$$

and dissipative in a similar sense to (6), so that, for some $\alpha_0 > 0$ and $\beta_0 \geq 0$

$$\langle F(x), x\rangle \leq -\alpha_0 |x|^2 + \beta_0,$$

we recover (6). Observe that, in this case, we are assuming that $f(t, x_t) = F(x(t-\rho(t))$ or, more generally, $f(t,\psi) = F(\psi(-\rho(t))$, for all $\psi \in \mathcal{C}, t \in \mathbb{R}$.

Indeed, we have

$$\begin{aligned}
\langle F(x(t-\rho(t)), x(t)\rangle &\leq \langle F(x(t)), x(t)\rangle + \langle F(x(t-\rho(t)) - F(x(t)), x(t)\rangle \\
&\leq -\alpha_0|x(t)|^2 + \beta_0 + |x(t)|\,|F(x(t-\rho(t)) - F(x(t))| \\
&\leq -\alpha_0|x(t)|^2 + \beta_0 + |x(t)|\,\omega(|x(t-\rho(t)) - x(t)|) \\
&\leq -\alpha_0|x(t)|^2 + \beta_0 + |x(t)|\omega(kh) \\
&\leq -\alpha|x(t)|^2 + \beta
\end{aligned}$$

for all $t \geq h$, since then $x(t)$ is a solution of (2).

We now show that (6) ensures the existence of a pullback attractor.

Theorem 4.2. *Suppose that (6) holds. Then there exists a family of bounded absorbing sets $\{B(t_0)\}_{t_0\in\mathbb{R}}$ for (3), and thus we can conclude the existence of a pullback attractor for this problem.*

Proof. We will prove more than the existence of a family of bounded absorbing sets: in fact, there exists a uniform (in t_0) bounded absorbing set for (3). Indeed, given $D \subset \mathcal{C}$ bounded, there exists $d \geq 0$ such that for all $\psi \in D$, $\|\psi\|_{\mathcal{C}} \leq d$, i.e.

$$\|\psi\|_{\mathcal{C}} = \sup_{\theta\in[-h,0]} |\psi(\theta)| \leq d.$$

Take now $\psi \in D$ and consider

$$\begin{aligned}
|\phi(t_0, t_0 - t)\psi| &= \sup_{\theta\in[-h,0]} |x(t_0+\theta; t_0 - t, \psi)| \\
&= \sup_{\tau\in[t_0-h,t_0]} |x(\tau; t_0 - t, \psi)|\,.
\end{aligned}$$

Let us write $x(\tau) = x(\tau; t_0 - t, \psi)$, $\tau \in [t_0 - t, t_0]$. Then, multiplying (5) by $x(\tau)$ we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|x(\tau)|^2 = 2\,\langle x(\tau), f(\tau, x_\tau)\rangle \leq 2\beta - 2\alpha|x(\tau)|^2,$$

for all $\tau \geq t_0 - t$.

Then, by Gronwall's lemma

$$\begin{aligned}
|x(\tau)|^2 &\leq |x(t_0 - t)|^2 e^{-2\alpha(\tau - t_0 + t)} + \frac{\beta}{\alpha}(1 - e^{-2\alpha(\tau - t_0 + t)}) \\
&\leq |\psi|^2 e^{-2\alpha(\tau - t_0 + t)} + \frac{\beta}{\alpha} \\
&\leq |\psi|^2 e^{-2\alpha(\theta + t)} + \frac{\beta}{\alpha} \\
&\leq de^{2\alpha h} e^{-2\alpha t} + \frac{\beta}{\alpha}.
\end{aligned}$$

Thus we obtain

$$\sup_{\theta \in [-h,0]} |x(t_0 + \theta)|^2 \leq de^{2\alpha h} e^{-2\alpha t} + \frac{\beta}{\alpha} \leq 1 + \frac{\beta}{\alpha}$$

if we take $t \geq \frac{1}{2\alpha} \log(de^{2\alpha h}) = T_D$. Note that this time T_D does not depend on t_0. ∎

4.2. A more general case

In the previous section we considered differential equations which only depended on the delay term, and had no explicit dependence of the current state (in other words, $f(t, x_t) = F(x(t), x(t - \rho(t))) = F(x(t - \rho(t))))$. However, a dependence on both the current and retarded state is more usual in applications, and often the equation can be interpreted as a perturbation of an ordinary differential equation.

In this section we will assume that F can be written as the following sum:

$$F(x(t), x(t - \rho(t))) = F_0(x(t)) + F_1(x(t - \rho(t))).$$

In this situation, we shall show that if a dissipative hypothesis for the term F_0 holds, then the assumptions on the other term can be relaxed.

Let us assume that $F_0 : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous functions satisfying the dissipative assumption as above:

$$\langle F_0(x), x \rangle \leq -\alpha_0 |x|^2 + \beta_0, \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{7}$$

Firstly, if we suppose that $F_1 : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous and bounded function, i.e. there exists $k \geq 0$ such that

$$|F_1(x)| \leq k, \quad \text{for all} \quad x \in \mathbb{R}^n,$$

then it is easy to prove that (6) holds. Indeed, for every $\psi \in \Phi(h)\mathcal{C}$ and a fixed $\varepsilon < \alpha_0$,

$$\begin{aligned}\langle f(t,\psi),\psi(0)\rangle &= \langle F_0(\psi(0)),\psi(0)\rangle + \langle F_1(\psi(-\rho(t))),\psi(0)\rangle \\ &\le -\alpha_0|\psi(0)|^2 + \beta_0 + k|\psi(0)| \\ &\le -(\alpha_0-\varepsilon)|\psi(0)|^2 + \beta_0 + \tfrac{k^2}{4\varepsilon}.\end{aligned}$$

Secondly, it is still possible to weaken this boundedness on F_1, although now it is necessary to assume more regularity for the delay function. Instead of proving that (6) holds, we prove the existence of a family of bounded absorbing sets directly.

THEOREM 4.3. *Assume F_0 satisfies (7). Assume that F_1 is sublinear, i.e. there exists $k > 0$ such that*

$$|F_1(x)|^2 \le k^2(1+|x|^2), \quad \text{for all } x \in \mathbb{R}^n,$$

and suppose that the delay function ρ is continuously differentiable with $\rho'(t) \le \rho_* < 1$. Then, if $k^2 < \alpha_0^2(1-\rho_*)$, there exists a family of bounded absorbing sets, $\{B(t_0)\}_{t_0\in\mathbb{R}}$ for (3), and consequently there exists a pullback attractor for this problem.

Proof. Choose a positive λ (small enough) and another positive ε which will be fixed later. As in the last theorem, let us write $x(\tau) = x(\tau; t_0-t,\psi)$, $\tau \in [t_0-t, t_0]$, for ψ in a given bounded set $D \subset \mathcal{C}$, i.e. $\|\psi\|_{\mathcal{C}} \le d$, for all $\psi \in D$. Then, it follows

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}\tau} e^{\lambda\tau}|x(\tau)|^2 &= \lambda e^{\lambda\tau}|x(\tau)|^2 + 2e^{\lambda\tau}\langle x(\tau), f(\tau,x_\tau)\rangle \\ &= \lambda e^{\lambda\tau}|x(\tau)|^2 + 2e^{\lambda\tau}\langle x(\tau), F_0(x(\tau))\rangle \\ &\quad +2e^{\lambda\tau}\langle x(\tau), F_1(x(\tau-\rho(\tau)))\rangle \\ &\le (\lambda-2\alpha_0)e^{\lambda\tau}|x(\tau)|^2 + 2\beta_0 e^{\lambda\tau} \\ &\quad +\varepsilon e^{\lambda\tau}|x(\tau)|^2 + e^{\lambda\tau}\varepsilon^{-1}\left|F_1(x(\tau-\rho(\tau)))\right|^2 \\ &\le (\lambda-2\alpha_0+\varepsilon)e^{\lambda\tau}|x(\tau)|^2 + (2\beta_0+k^2\varepsilon^{-1})e^{\lambda\tau} \\ &\quad +k^2\varepsilon^{-1}e^{\lambda\tau}\left|x(\tau-\rho(\tau))\right|^2.\end{aligned}$$

By integration on the interval $[t_0-t,\tau]$,

$$\begin{aligned}e^{\lambda\tau}|x(\tau)|^2 - e^{\lambda(t_0-t)}|x(t_0-t)|^2 \le\ & \tfrac{2\beta_0+k^2\varepsilon^{-1}}{\lambda}\left[e^{\lambda\tau} - e^{\lambda(t_0-t)}\right] \\ &+(\lambda-2\alpha_0+\varepsilon)\int_{t_0-t}^{\tau} e^{\lambda s}|x(s)|^2 ds \\ &+\tfrac{k^2}{\varepsilon}\int_{t_0-t}^{\tau} e^{\lambda s}|x(s-\rho(s))|^2 ds.\end{aligned} \tag{8}$$

Evaluating the term containing the delay function by making the change of variable $s-\rho(s)=u$ in the integral, we obtain

$$\begin{aligned}\int_{t_0-t}^{\tau} e^{\lambda s}|x(s-\rho(s))|^2 ds &\le \tfrac{1}{1-\rho_*}\int_{t_0-t-h}^{\tau} e^{\lambda u+\lambda h}|x(u)|^2 du\\ &\le \tfrac{e^{\lambda h}}{1-\rho_*}\left[\int_{t_0-t-h}^{t_0-t} e^{\lambda u}|x(u)|^2 du+\int_{t_0-t}^{\tau} e^{\lambda u}|x(u)|^2 du\right]\\ &\le \tfrac{e^{\lambda h}}{1-\rho_*}\left[\int_{t_0-t-h}^{t_0-t} e^{\lambda u}|\psi(u)|^2 du+\int_{t_0-t}^{\tau} e^{\lambda u}|x(u)|^2 du\right]\\ &\le \tfrac{e^{\lambda h}}{1-\rho_*}\int_{t_0-t}^{\tau} e^{\lambda u}|x(u)|^2 du\\ &\quad+\tfrac{d^2e^{\lambda h}}{\lambda(1-\rho_*)}\left[e^{\lambda(t_0-t)}-e^{\lambda(t_0-t-h)}\right],\end{aligned}$$

and, consequently,

$$\begin{aligned}e^{\lambda\tau}|x(\tau)|^2 &\le e^{\lambda(t_0-t)}d^2+\left(2\beta_0+k^2\varepsilon^{-1}\right)\lambda^{-1}\left[e^{\lambda\tau}-e^{\lambda(t_0-t)}\right]\\ &\quad+\tfrac{d^2e^{\lambda h}k^2\varepsilon^{-1}}{\lambda(1-\rho_*)}\left[e^{\lambda(t_0-t)}-e^{\lambda(t_0-t-h)}\right]\\ &\quad+\left[\lambda-2\alpha_0+\varepsilon+\tfrac{e^{\lambda h}k^2\varepsilon^{-1}}{(1-\rho_*)}\right]\int_{t_0-t}^{\tau} e^{\lambda s}|x(s)|^2 ds.\end{aligned}$$

Now, taking $\varepsilon=\alpha_0$ and noticing that for λ small enough we can assure that $\lambda-2\alpha_0+\varepsilon+\frac{e^{\lambda h}k^2\varepsilon^{-1}}{(1-\rho_*)}$ is negative, it immediately follows that

$$|x(\tau)|^2 \le d^2\left[1+\tfrac{e^{\lambda h}k^2\varepsilon^{-1}}{\lambda(1-\rho_*)}\right]e^{\lambda(t_0-t-\tau)}+\left(2\beta_0+k^2\varepsilon^{-1}\right)\lambda^{-1},$$

and setting $\tau=t_0+\theta$, for $\theta\in[-h,0]$,

$$|x(t_0+\theta)|^2\le d^2\left[1+\frac{e^{\lambda h}k^2\varepsilon^{-1}}{\lambda(1-\rho_*)}\right]e^{-\lambda(t+\theta)}+\left(2\beta_0+k^2\varepsilon^{-1}\right)\lambda^{-1},$$

and, thus

$$\begin{aligned}\sup_{\theta\in[-h,0]}|x(t_0+\theta)|^2 &\le d^2\left[1+\tfrac{e^{\lambda h}k^2\varepsilon^{-1}}{\lambda(1-\rho_*)}\right]e^{-\lambda t+\lambda h}+\left(2\beta_0+k^2\varepsilon^{-1}\right)\lambda^{-1}\\ &\le 1+\left(2\beta_0+k^2\varepsilon^{-1}\right)\lambda^{-1}\end{aligned}$$

if

$$t\ge T_D=\lambda^{-1}\log d^2\left[1+\frac{e^{\lambda h}k^2\varepsilon^{-1}}{\lambda(1-\rho_*)}\right]e^{\lambda h}.$$

∎

Remark 4.1. Notice that we have chosen the best ε in the proof of the theorem. Indeed, as

$$\lim_{\lambda \downarrow 0} \left[\lambda - 2\alpha_0 + \varepsilon + \frac{e^{\lambda h} k^2 \varepsilon^{-1}}{(1-\rho_*)} \right] = -2\alpha_0 + \varepsilon + \frac{k^2 \varepsilon^{-1}}{(1-\rho_*)},$$

this value will be negative iff

$$\frac{k^2 \varepsilon^{-1}}{(1-\rho_*)} < 2\alpha_0 - \varepsilon \Leftrightarrow k^2 < \varepsilon(2\alpha_0 - \varepsilon)(1-\rho_*),$$

and the maximum of the function $v(\varepsilon) = \varepsilon(2\alpha_0 - \varepsilon)$ is achieved at $\varepsilon = \alpha_0$.

4.3. The case of weak dissipativity

We now prove a similar result, but in this case the dissipativity condition need not be uniform in t. We suppose that $f(\cdot,\cdot) : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ satisfies the following three conditions, where $\gamma_i(t)$ are positive continuous functions:

(i) A Lipschitz condition, local in time,

$$|f(t,\psi_1) - f(t,\psi_2)| \le \gamma_1(t) \|\psi_1 - \psi_2\|_{\mathcal{C}}, \qquad \text{for all} \qquad \psi_1, \psi_2 \in \mathcal{C}, \tag{9}$$

(ii) A local dissipativity condition,

$$\langle f(t,\psi), \psi(0) \rangle \le (-\alpha + \gamma_1(t)) |\psi(0)|^2 + \gamma_2(t), \text{ for all } \psi \in \Phi(h)\mathcal{C}, \tag{10}$$

and

(iii) Some integrability conditions,

$$\int_{-\infty}^{t} \gamma_1(s) ds < \infty \qquad \text{and} \qquad \int_{-\infty}^{t} e^{\varepsilon s} \gamma_2(s) ds < \infty, \text{ for all } \varepsilon > 0. \tag{11}$$

We now show, as in section 4.1, that these conditions can be derived for $F(t, x(t-\rho(t)))$ given similar properties of $F(t,x)$. Of course, these conditions could be relaxed following the arguments in section 4.2, but to avoid too many technical computations we content ourselves with the consideration of the case below.

We note that (9–11) are satisfied by f if, for example, we impose on F in (2) the following conditions

$$\begin{aligned} |F(t,x) - F(t,y)| &\le \gamma_1(t)|x-y| \\ \langle F(t,x), x \rangle &\le (-\alpha + \gamma_1(t))|x|^2 + \gamma_2(t), \end{aligned}$$

with $\gamma_i(t)$ as above. Additionally, once again we require F to be bounded, $|F(x)| \leq k$.

Then,

$$\begin{aligned}
\langle F(t, x(t-\rho(t))), x(t)\rangle &\leq (-\alpha + \gamma_1(t))|x(t)|^2 + \gamma_2(t) \\
&\quad + \langle F(t, x(t)) - F(t, x(t-\rho(t))), x(t)\rangle \\
&\leq (-\alpha + \gamma_1(t))|x(t)|^2 + \gamma_2(t) \\
&\quad + \gamma_1(t)|x(t) - x(t-\rho(t))||x(t)| \\
&\leq (-\alpha + \gamma_1(t))|x(t)|^2 + \gamma_2(t) + \gamma_1(t)kh|x(t)| \\
&\leq (-\alpha + \gamma_1(t))|x(t)|^2 + \gamma_2(t) \\
&\quad + \frac{1}{2}|x(t)|^2\gamma_1(t) + \frac{1}{2}(kh)^2\gamma_1(t) \\
&\leq (-\alpha + \frac{3}{2}\gamma_1(t))|x(t)|^2 + \frac{1}{2}(kh)^2\gamma_1(t) + \gamma_2(t).
\end{aligned}$$

Under (9)–(11), equation (3) is well-posed, and as before we denote by $x(t; s, \psi)$ the value at time t of the unique solution to (3) with $x_s = \psi \in \mathcal{C}$:

$$\phi(t, s)\psi(\theta) = x(t + s + \theta; s, \psi)$$

THEOREM 4.4. *Under the above conditions, there exists a pullback attractor for problem (3).*

The proof also shows that under these conditions, in general there does not exist an absorbing set for the evolution forwards in time.

Proof. We will prove the existence of a time-dependent bounded absorbing ball. It is not difficult to check that for $t \geq s$, $s \in \mathbb{R}$, and $\psi \in \mathcal{C}$,

$$\phi(s, s-t)\psi(\theta) = x(s + \theta; -t + s, \psi).$$

Now, setting $x(\tau) = x(\tau; -t + s, \psi)$ for $\tau \geq -t + s$, and taking the scalar product in the equation (3) with $x(\tau)$, by virtue of (10), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}|x(\tau)|^2 \leq -\alpha|x(\tau)|^2 + \gamma_1(\tau)|x(\tau)|^2 + \gamma_2(\tau)$$

and, therefore

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|x(\tau)|^2 \leq -2\alpha|x(\tau)|^2 + 2\gamma_1(\tau)|x(\tau)|^2 + 2\gamma_2(\tau)$$

Applying Gronwall's lemma in the interval $[-t+s,\tau]$, it follows that

$$\begin{aligned}
|x(\tau)|^2 &\leq \|\psi\|_{\mathcal{C}}^2 \exp\left(\int_{-t+s}^{\tau}(-\alpha+2\gamma_1(r))dr\right) \\
&\quad +2\int_{-t+s}^{\tau}\gamma_2(r)\exp\left(\int_r^{\tau}(-\alpha+2\gamma_1(\xi))d\xi\right)dr \\
&\leq \|\psi\|_{\mathcal{C}}^2 e^{\alpha h}\exp\left(\int_{-t+s}^{s}2\gamma_1(r)dr\right)e^{-\alpha t} \\
&\quad +2\int_{-\infty}^{s}\gamma_2(r)\exp\left(-\alpha(s-h-r)+\int_{-\infty}^{s}2\gamma_1(\xi)d\xi\right)dr \\
&\leq \|\psi\|_{\mathcal{C}}^2 e^{M_s}e^{-\alpha t}+2e^{\alpha h+2M_s}\int_{-\infty}^{s}\gamma_2(r)e^{\alpha r}dr,
\end{aligned}$$

with $M_s=\int_{-\infty}^{s}\gamma_1(\xi)d\xi$. Thus, if we set

$$r^2(s)=1+2e^{\alpha h+2M_s}\int_{-\infty}^{s}\gamma_2(r)e^{\alpha r}dr,$$

it is clear that $\overline{B_{\mathcal{C}}}(0,r(0))$ is a bounded absorbing set for the cocycle $\phi(t,s)$ associated to (3). Theorem 3.1 now ensures the existence of a pullback attractor $\mathcal{A}(s)$ for (3). ▮

5. PULLBACK ATTRACTORS FOR PERIODIC EQUATIONS

In this section we compare our results with those in Hale [7] and Hale and Verduyn Lunel [8], and show that when we have a periodic nonlinear term we recover their results: the pullback attractor reduces to a periodic uniform forward attractor.

Suppose there exists $T>0$ such that

$$f(t+T,\psi)=f(t,\psi),\ \text{for all } t\in\mathbb{R},\ \psi\in\mathcal{C}.$$

Then it is not difficult to prove that the process $\phi(t,s)$ is also periodic. Indeed, if we define $X(t)=x(t+T;s+T,\psi)$, it satisfies

$$\begin{aligned}
\frac{\mathrm{d}X(t)}{\mathrm{d}t} &= \frac{\mathrm{d}x(t+T)}{\mathrm{d}(t+T)}=\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} \\
&= f(\tau,x_\tau) \\
&= f(t+T,x_\tau) \\
&= f(t,X_t),
\end{aligned}$$

and so we get that

$$\phi(t+T,s+T)\psi=\phi(t,s)\psi.$$

From this expression we also conclude the periodicity of the pullback attractor, since, for the omega limit set of any bounded $D \subset \mathcal{C}$ we have that

$$\begin{aligned}\Lambda_D(t_0) &= \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}\phi(t_0,t_0-t)D} \\ &= \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}\phi(t_0+T,t_0+T-t)D} \\ &= \Lambda_D(t_0+T),\end{aligned}$$

and then

$$\mathcal{A}(t_0) = \overline{\bigcup_{\substack{D\subset\mathcal{C}\\ \text{bounded}}}\Lambda_D(t_0)} = \overline{\bigcup_{\substack{D\subset\mathcal{C}\\ \text{bounded}}}\Lambda_D(t_0+T)} = \mathcal{A}(t_0+T).$$

Furthermore, from the attraction property

$$\lim_{t\to+\infty}\operatorname{dist}(\phi(t_0,t_0-t)D,\mathcal{A}(t_0)) = 0, \tag{12}$$

and the periodicity of $\mathcal{A}(\cdot)$, we can conclude an uniform pullback attraction to $\widetilde{\mathcal{A}} = \underset{t\in[0,T]}{\cup}\phi(t,0)\mathcal{A}(0)$. Indeed, from (12),

$$\lim_{t\to+\infty}\sup_{0\leq t_0\leq T}\operatorname{dist}(\phi(t_0,t_0-t)D,\widetilde{\mathcal{A}}) = 0$$

and then

$$\lim_{t\to+\infty}\sup_{0\leq t_0\leq T}\sup_{k\in\mathbb{Z}}\operatorname{dist}(\phi(t_0+kT,t_0+kT-t)D,\widetilde{\mathcal{A}}) = 0,$$

so that

$$\lim_{t\to+\infty}\sup_{t_0\in\mathbb{R}}\operatorname{dist}(\phi(t_0,t_0-t)D,\widetilde{\mathcal{A}}) = 0. \tag{13}$$

But the uniform pullback convergence in (13) implies uniform forward convergence to $\widetilde{\mathcal{A}}$, since, for $\tau = t_0 - t$ (t fixed)

$$\sup_{t_0\in\mathbb{R}} dist(\phi(t_0,t_0-t)D,\widetilde{\mathcal{A}}) = \sup_{\tau\in\mathbb{R}} dist(\phi(\tau+t,\tau)D,\widetilde{\mathcal{A}}),$$

and thus

$$\lim_{t\to+\infty}\sup_{\tau\in\mathbb{R}}\operatorname{dist}(\phi(\tau+t,\tau)D,\widetilde{\mathcal{A}}) = 0.$$

Note that $\widetilde{\mathcal{A}}$ coincides with the global attractor obtained in Hale [7] (Theorem 4.1.11). Indeed, he writes $M \subset \mathbb{R} \times \mathcal{C}$ for the global attractor associated to (3), which is, in our case, precisely the set

$$\mathbb{A} = \{(t_0, \psi) : \psi \in \mathcal{A}(t_0),\ t_0 \in \mathbb{R}\}.$$

6. EQUATIONS WITH 'SMALL' DELAYS

Now consider a family of retarded differential equations parametrised by ϵ,

$$\dot{x} = F(x(t - \rho_\epsilon(t))) \tag{14}$$

for which the delay $\rho_\epsilon(t)$ is constrained to lie within an interval $[0, \epsilon]$, and where (for the sake of this discussion) we assume (cf. section 4.1.1) that

$$\langle F(x), x \rangle \leq -\alpha |x|^2 + \beta \tag{15}$$

along with global boundedness and Lipschitz conditions for F,

$$|F(x)| \leq k \qquad \text{and} \qquad |F(x) - F(y)| \leq L|x - y| \tag{16}$$

It is natural to consider the relationship between the pullback attractor which we can find for $\epsilon > 0$, and the standard global attractor obtained when $\epsilon = 0$.

To this end, we recall a general result from Caraballo and Langa [2].

Theorem 6.1. *Let ϕ_ϵ be a family of processes on a space X. Suppose that the following conditions are satisfied:*

1.Existence of attractors for $\epsilon > 0$: for $\epsilon \in (0, \epsilon_0]$ there exists a pullback attractor $\{\mathbb{A}_\epsilon(t)\}$,

2.Convergence of processes to a semiflow: for each $s \in \mathbb{R}$, $t \in [0, \infty)$ we have

$$\phi_\epsilon(s + t, s)x \to \mathcal{S}(t)x \quad \text{as} \quad \epsilon \downarrow 0,$$

uniformly for x in bounded sets of X,

3.Existence of an attractor for the semiflow: $\mathcal{S}(t)$ has an attractor $\mathbb{A}$

4.Compactness property: there exists a compact set K such that, for each t,

$$\lim_{\epsilon \to 0} \operatorname{dist}(\mathbb{A}_\epsilon(t), K) = 0.$$

Then for each t,

$$\lim_{\epsilon \to 0} \operatorname{dist}(\mathbb{A}_\epsilon(t), \mathbb{A}) = 0. \tag{17}$$

Note that the comparison of the attractors in the theorem occurs in one fixed phase space X. Therefore, in order to apply the theorem to (14), we choose one ϵ_0, and consider the equation on the phase space

$$\mathcal{C}_0 = C^0([-\epsilon_0, 0]; \mathbb{R}^n)$$

whatever the value of $\epsilon \in [0, \epsilon_0]$. In particular, we have to consider the attractor of the autonomous equation

$$\dot{x} = F(x) \tag{18}$$

as a collection of functions in $\mathcal{C}_0$.

If we denote the semigroup on $\mathbb{R}^n$ corresponding to (18) as $S(t)$, then clearly, if $\mathcal{A}$ is the usual global attractor in $\mathbb{R}^n$ for (18) the attractor in $\mathcal{C}_0$ is given by

$$\mathbb{A} = \{x \in \mathcal{C}_0 : x(t) = S(t)u_0, t \in [-\epsilon_0, 0], u_0 \in \mathcal{A}\} \tag{19}$$

Furthermore, the semigroup $\mathcal{S}(t)$ on $\mathcal{C}_0$ generated by (18) is given by

$$[\mathcal{S}(t)\psi](s) = S(t)\psi(s).$$

In this sense, we prove the following theorem.

THEOREM 6.2. *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies (15) and (16).Then, for every $0 < \epsilon \leq \epsilon_0$ there exists a pullback attractor $\{\mathbb{A}_\epsilon(t)\}$ for (14) in the space $\mathcal{C}_0$. Furthermore, $\dot{x} = F(x)$ has a global attractor in $\mathcal{C}_0$ in the sense of (19), and the upper semicontinuity property (17) holds*

Of course, (17) holds with the distance measured in $\mathcal{C}_0$.

Proof. We discuss the four conditions of theorem 6.1 in order.

1. This is simply theorem 4.2 for each $\epsilon > 0$. Adapting the result to treat the longer time interval $[-\epsilon_0, 0]$ for $\epsilon < \epsilon_0$ is essentially trivial.

2. We define $\mathcal{S}(t)$, as above, to be the semiflow on $\mathcal{C}_0$ arising from the autonomous problem

$$\dot{x} = F(x),$$

and compare the solution of this with

$$\dot{y} = F(y(t - \rho(t))).$$

Given an initial condition $x_s = \psi$, with $\|\psi\|_{\mathcal{C}_0} \le M$, consider

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) - y(t)) = F(x(t)) - F(y(t - \rho(t))).$$

Taking the inner product with $x(t) - y(t)$ gives

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t) - y(t)|^2 &= \langle F(x(t)) - F(y(t)), x(t) - y(t)\rangle \\
&\qquad + \langle F(y(t)) - F(y(t - \rho(t))), x(t) - y(t)\rangle \\
&\le L|x(t) - y(t)|^2 + L|y(t) - y(t - \rho(t))||x(t) - y(t)| \\
&\le 2L|x(t) - y(t)|^2 + \begin{cases} 2M & 0 \le t \le \epsilon \\ kL|\rho(t)| & t \ge \epsilon \end{cases} \\
&\le 2L|x(t) - y(t)|^2 + \begin{cases} 2M & 0 \le t \le \epsilon \\ kL\epsilon & t \ge \epsilon. \end{cases}
\end{aligned}$$

On $[0, \epsilon]$ we can deduce that

$$|x(t) - y(t)|^2 \le (e^{Lt} - 1)\frac{2M}{L},$$

and so, in particular,

$$|x(t) - y(t)|^2 \le (e^{L\epsilon} - 1)\frac{2M}{L} \qquad \text{for all} \qquad t \in [0, \epsilon].$$

Now, starting from $t = \epsilon$ we have

$$|x(t) - y(t)|^2 \le \frac{1}{L}[2M(e^{L\epsilon} - 1) + kL\epsilon]e^{L(t-\epsilon)},$$

and, therefore,

$$|x(t) - y(t)|^2 \le C(\epsilon, t),$$

where $C(\epsilon, t) \to 0$ as $\epsilon \downarrow 0^+$ uniformly on bounded time intervals. In particular, it follows that

$$\sup_{s \in \mathbb{R}} \|\phi(t + s, s)\psi - \mathcal{S}(t)\psi\|_{\mathcal{C}_0} \to 0 \quad \text{for all} \quad t \ge 0. \tag{20}$$

3. This follows immediately from (15), since the proof of the existence of an absorbing set in $\mathbb{R}^n$ under this condition is simple. There is then a global attractor $\mathcal{A} \subset \mathbb{R}^n$, and we can define $\mathbb{A}$ as in (19).

4. Finally, note that the radius of the absorbing set B in theorem 4.2 depends on α and β, and using the calculations from section 4.1.1 it follows

that α and β can be taken uniform over $\epsilon \in (0, \epsilon_0]$. It then follows that the compact absorbing set in theorem 4.2 is given by

$$\phi_\epsilon(t_0, t_0 - \epsilon_0)B.$$

It follows from (20) that for every $\epsilon \in (0, \epsilon_0]$ this is a subset of a fixed compact set K. Since $\mathbb{A}_\epsilon(t) \subset B_\epsilon(t)$ (see theorem 3.1) the final condition is satisfied, and an application of theorem 6.1 gives the result as stated. ▮

Remark 6.1. Note that one could also compare the attractors by considering the subsets of $\mathbb{R}^n$

$$\mathcal{A}_\epsilon(t) = \{y \in \mathbb{R}^n : \; y = x(t), \; t \in [-\epsilon, 0], \; x \in \mathbb{A}_\epsilon(t)\}.$$

It then follows that

$$\mathrm{dist}(\mathcal{A}_\epsilon(t), \mathcal{A}) \to 0$$

as $\epsilon \to 0$, where now the distance is measured in $\mathbb{R}^n$.

7. CONCLUSIONS

By using the cocycle attractor we have extended the classical treatment of attractors for delay differential equations to the general nonautonomous case, and in particular we have recovered results on periodic equations. Furthermore, by using the upper semicontinuity result from Carballo and Langa [2], we have shown that the introduction of a small delay has little effect on the asymptotic dynamics.

Finally, we note that these ideas should be applicable to a wider class of equations. In particular, equations with distributed delays, such as

$$\dot{x}(t) = \int_{-h}^{0} f(t, s, x(t+s))\, ds,$$

and even equations with infinite delays, such as

$$\dot{x}(t) = \int_{-\infty}^{0} f(t, s, x(t+s))\, ds.$$

For the second of these the phase space needs to be chosen much more carefully than above, and we have not presented results for this system here to avoid too much notation. For details of the standard theory see Kuang [11].

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